MATH 2068: Honours Mathematical Analysis II: Home Examination
7:00 pm, 29 April 2022

Important Notice:

♣ The answer paper **must be submitted before 30 April 2022 at 7:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ **The answer paper must include your name and student ID.**

Answer ALL Questions

1. **(25 points)**

   (i) Does the series $f(x) := \sum_{k=1}^{\infty} ke^{-kx}$ converge uniformly on its convergence domain $D := \{x \in \mathbb{R} : f(x) \text{ is convergrent}\}$? In addition, is f continuous on D?

   (ii) Let (a_n) and (b_n) be two sequences of non-negative numbers. Prove or disprove the following statements:

   (a) If $\sum a_n$ is convergent and $\lim na_n$ exists, then $\lim na_n = 0$.

   (b) Assume that $b_n > 0$ for all n and $\lim \frac{a_n}{b_n} = 0$. If $\sum b_n$ is convergent then so is $\sum a_n$.

   (iii) Let (x_n) and (y_n) be the sequences of numbers. Assume that we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |y_n x_k| = 0 \tag{1}$$

   Show that $\lim_{k\to\infty} \sup\{|y_n x_k| : n = 1, 2, ...\} = 0$.

2. **(25 points)**

Let h be a non-negative real-valued function defined on $\mathbb{R}$. Suppose that h is Riemann integrable over every closed and bounded interval. For each pair of numbers a and b set $\tau(a,a) = 0$ and if $a \neq b$, set

$$\tau(a,b) := |\int_a^b h(t)dt|. \tag{2}$$

(i) Let a and b be a pair of numbers with $a < b$. Show that for any $t \in [0,1]$, there is a real number x_t such that $\tau(a, x_t) = t\tau(a,b)$ and $\tau(x_t, b) = (1-t)\tau(a,b)$.

(ii) Let f a bounded function defined on $\mathbb{R}$ satisfying the condition: if for each $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbb{R}$ with $\tau(x,y) < \delta$. Show that for each $L > 0$, there is $C > 0$ such that $\tau(f(x), f(y)) \leq C\tau(x,y)$ as $\tau(x,y) > L$.

(iii) Now let f be a bounded uniform continuous on $\mathbb{R}$, that is $h(t) \equiv 1$ in Eq(2). Prove or disprove the following statement:
If f satisfies the condition: for each $L > 0$, there is $C > 0$ such that $|f(x) - f(y)| \leq C|x-y|$ as $|x-y| > L$, then f is a Lipschitz function, that is there is $C > 0$ such that $|f(x) - f(y)| \leq C|x-y|$ for all $x, y \in \mathbb{R}$.

$* * *$ **END OF PAPER** $* * *$